

# The Effects of the modified scalar product on some properties of the one-dimensional harmonic oscillator with energy-dependent potential

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In this article, we try to test the influence of the modification of the scalar product, found in the problems of the energy-dependent potential, on the physical properties of the harmonic oscillator in one dimension. For this, we at first discuss the effect of this change on the thermodynamic properties of this oscillator, and then on the parameters of Fisher and Shannon of quantum information. For the second problem, we are an obligation to redefine this parameters. Finally, the uncertainly relation of Cramer-Rao is well recovered in our problem in question.

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## I. INTRODUCTION

Wave equations with energy dependent potentials have been come to view for long time. They can be visible in Klein-Gordon equation considering particle in an external electromagnetic field[1]. Arising from momentum dependent interactions, they also can be appear in non-relativistic quantum mechanics, as shown by Green [2] for instance Pauli-Schrodinger equation possess another example [3, 4]. Sazdjian [5] and Formanek et al [6] have noted that the density probability, or the scalar product, has to be modified with respect to the usual definition, in order to have a conserved norm. Garcia-Martinez et al and Lombard made an investigation on Schrödinger equation with energy-dependent potentials by solving them exactly in one and three dimensional [7, 8] as well as Hassanabadi et al studied D-dimensional Schrodinger equation for an energy-dependent Hamiltonian that linearly depends on energy and quadratic on the relative distance [9] and also they studied in another case Dirac equation for an energy-dependent potential in the presence of spin and pseudospin symmetries with arbitrary spin-orbit quantum number . We calculate the corresponding eigenfunctions and eigenvalues of this system by comparing with analytically solvable energy-dependent potentials [10]. As an example of a many-body problem with an energy-dependent potential, we can mention what was done by Lombard and Mareš [11]. They considered systems of N bosons bound by two-body harmonic interactions, whose frequency depends on the total energy of the system and there are an interesting efforts in this literature in (see Refs. [12–15]).

The presence of the energy dependent potential in a wave equation has several non-trivial implications. The most obvious one is the modification of the scalar product, necessary to ensure the conservation of the norm. This modification can modified some behavior or a physical properties of a physical system: here we note that this question, in best of our knowledge, has not been considered in the literature. In this context, the main goal of this paper is to study the effects of the modified scalar product of the wave function with potential dependent energy of a typical system such as the one-dimensional Harmonic oscillator. For this, we are focused on the study of: (i) the thermal properties of the 1D harmonic oscillator and, (ii) the Fisher and Shannon parameters of quantum information. This choice is justified by the dependence of these two parameters with the density of states. We propose in this article that in Sec. 2 harmonic oscillator with an energy-dependent frequency has been considered and discussed by deriving eigenfunctions, values density of a state at first. Then the effect of modified scalar product on the thermal properties will be the subject of the sec. III. In sec. IV, we, also, study such effects but on the Fisher and Shannon parameters of quantum information. Sec. V will be a conclusion.

## II. ONE-DIMENSIONAL HARMONIC OSCILLATOR WITH AN ENERGY-DEPENDENT POTENTIAL

### A. The eigensolutions

We consider the harmonic oscillator potential with an energy-dependent frequency

$$H = \frac{p^2}{2m} + \frac{m\omega^2(1 + \gamma E^\nu)}{2}x^2. \quad (1)$$

where  $\gamma$  is a parameter (not necessarily small, though most of our investigations are dedicated to small  $\gamma$ ). This Hamiltonian can be written in  $\{x\}$  representation as follows:

$$\hat{H} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{m\omega^2(1 + \gamma E^\nu)}{2}x^2, \quad (2)$$

with in the position representation the wave function is a function of  $x$ , i.e., is given by the function  $\psi(x)$ ; the momentum and position operators are

$$p \rightarrow -i\hbar \frac{d}{dx}, \quad x \rightarrow \hat{x}, \quad (3)$$

Consequently, the time-independent Schrodinger equation for the oscillator can be written

$$\left\{ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{m\omega^2(1 + \gamma E^\nu)}{2}x^2 \right\} \psi(x) = E\psi(x). \quad (4)$$

with the following substitutions,

$$\xi = \sqrt{\frac{2m\omega\sqrt{1 + \gamma E^\nu}}{\hbar}}x, \quad \varepsilon = \frac{E}{\hbar\omega\sqrt{1 + \gamma E^\nu}}, \quad \phi(\xi) = \psi\{x(\xi)\} \quad (5)$$

we obtain

$$\left\{ \frac{\partial^2}{\partial \xi^2} + \varepsilon - \frac{\xi^2}{4} \right\} \phi(\xi) = 0. \quad (6)$$

The last equation has the same form as

$$y''(z) + \left\{ \vartheta + \frac{1}{2} - \frac{z^2}{4} \right\} \phi(z) = 0, \quad (7)$$

where the solutions are

$$y(z) = C_1 D_\vartheta(z) + C_2 D_{-\vartheta-\frac{1}{2}}(iz). \quad (8)$$

Here  $D_\vartheta(z)$  are the Weber Hermite functions,  $C_{1,2}$  are the constants and  $z$  and  $\nu$  are complex numbers. In order that the solution  $y(z)$  can be interpreted as wave function, it must to check  $\lim_{\beta \rightarrow \pm\infty} |y(z)|^2 = 0$ , for a real argument of  $z$ . Only the function  $D_\nu(z)$  has this property if  $\nu$  is an integer. In that case

$$D_\vartheta(z) = e^{-\frac{z^2}{2}} He_\vartheta(z), \quad \vartheta = 0, 1, 2, \dots \quad (9)$$

where

$$He_\vartheta(z) = (-1)^\vartheta e^{\frac{z^2}{2}} \frac{d^\vartheta}{dz^\vartheta} e^{-\frac{z^2}{2}}, \quad (10)$$

are related to the Hermite polynomials  $H_\nu(z)$  via

$$He_\vartheta(z) = \sqrt{2}^{-\vartheta} H_\vartheta(\sqrt{2}z). \quad (11)$$

Finally, the eigensolutions are

$$\psi_n(x) = C_n e^{-\frac{m\omega\sqrt{1+\gamma E^\nu}x^2}{2\hbar}} H_n\left(\sqrt{\frac{m\omega\sqrt{1+\gamma E^\nu}}{\hbar}}x\right), \quad (12)$$

$$E^2 - \hbar^2\omega^2\left(n + \frac{1}{2}\right)^2 \gamma E^\nu - \hbar^2\omega^2\left(n + \frac{1}{2}\right)^2 = 0. \quad (13)$$

Now, Let us consider the following two particular cases (we use that  $\hbar = \omega = m = 1$ , and  $\gamma < 0$ ):

- First case:  $\nu = 1$ , and  $\omega^2 = 1 + \gamma E$ : the eigenvalues are

$$E_{1n} = \frac{\gamma}{8} (2n+1)^2 \pm \left(n + \frac{1}{2}\right) \sqrt{1 + \frac{\gamma^2}{16} (2n+1)^2}, \quad (14)$$

- Second case:  $\nu = 2$ , and  $\omega^2 = 1 + \gamma E^2$ : so we obtain

$$E_{2n} = \pm \frac{2n+1}{\sqrt{4 - \gamma(2n+1)^2}}. \quad (15)$$

The corresponding wave function in both cases is written in compact form by

$$\psi_{\alpha n}(x) = C_{\alpha n} e^{-\frac{\lambda_\alpha x^2}{2}} H_n\left(\sqrt{\lambda_\alpha}x\right), \quad (16)$$

with

$$\lambda_\alpha = \sqrt{1 + \gamma E_{\alpha n}^\nu}, \quad \text{where } \alpha = (1, 2). \quad (17)$$

In both cases, only the positive energies are retained, since the negative ones are not normalizable.

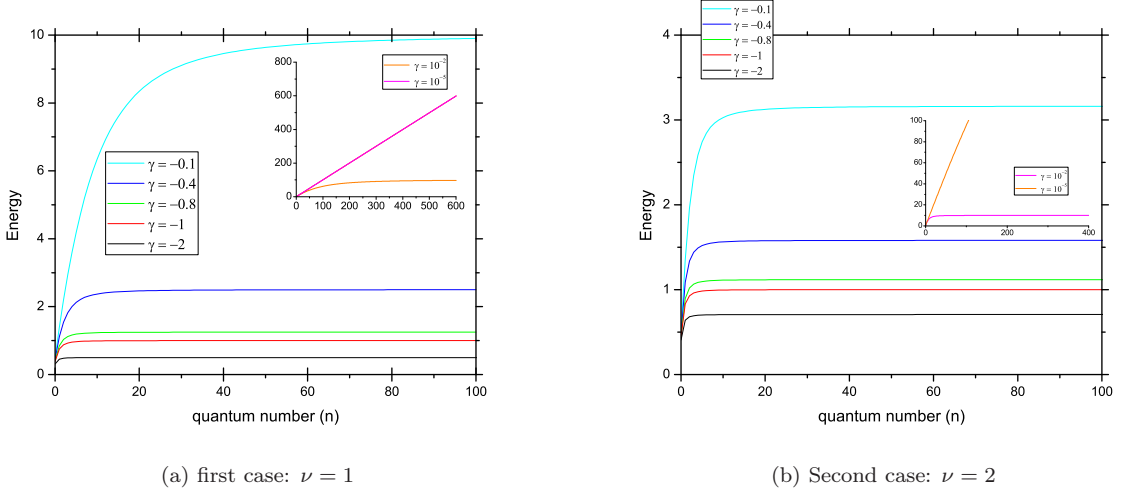


Figure 1: Spectrum of energy  $E$  versus quantum number ( $n$ ) for both cases.

Now we are in the main goal of our study, i.e., the influence of the modified scalar product on the wave function with energy-dependent potential: especially on the properties of eigenvalues. According to the works of [5, 6, 8], the definition of the density has to be modified in order to ensure the validity of the continuity equation. So, for the non-relativistic Schrodinger equation, the new definition for the density of a state  $n$  is given by

$$\rho_n(x) = |\psi_n(x)|^2 \left( 1 - \frac{dV}{dE} \right). \quad (18)$$

In order to represent a physical system, the density has to be positive definite, which means here

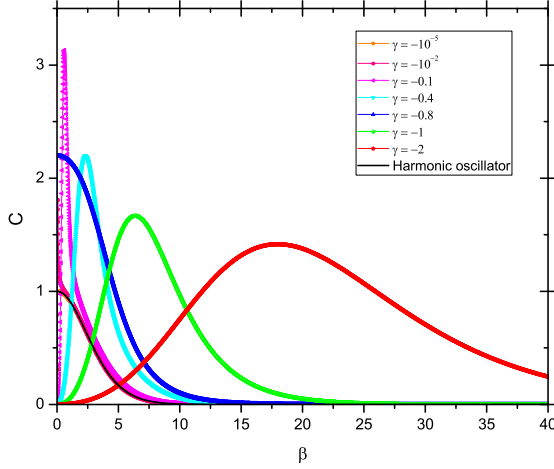
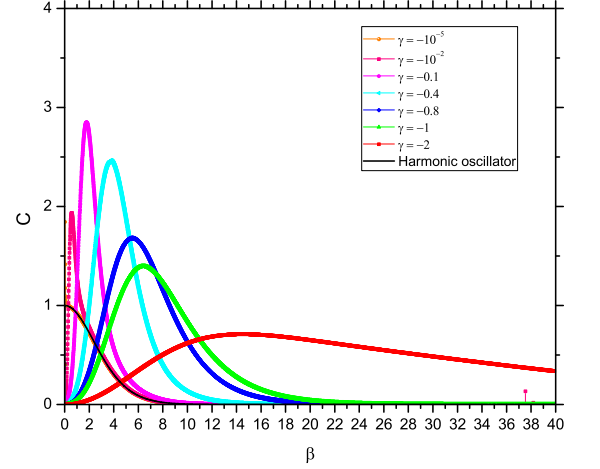
$$1 - \frac{dV}{dE} \geq 0. \quad (19)$$

This imposes constraints on the energy dependence for the theory to be coherent: by this, we mean a theory that have the following properties: (i) the necessary modification of the definition of probability density, (ii) The vectors corresponding to stationary states with different energies must be orthogonal, (iii) The formulation of the closure rule in terms of wave functions of stationary states justifies their standardization, (iv) finally, the operators of observables are all self-adjoint (Hermitian)[8, 12].

The condition that  $\rho_n(x) > 0$ , leads to the following implication that  $\gamma < 0$  for both cases. Now, we are ready to discuss some interesting results that are not well comments in the literature. Analytically, the asymptotic limits for both form of energies are  $\frac{1}{|\gamma|}$  for the first case, and  $\frac{1}{\sqrt{|\gamma|}}$  for the second one. These limits have been reproduced for both cases in Figure . 1 . Following this figure, some remarks can be made:

- the modified scalar product is the origin of that the spectrum exhibits saturation instead of growing infinitely,
- the analytical asymptotic limits are well depicted,
- the beginning of the saturation starts from a specific quantum number  $N$ ,
- in the limit where  $\gamma > -2$ , the saturation appears for all values of quantum number  $n$ .
- finally, when  $\gamma$  takes a very small values (in our case  $|\gamma| = 10^{-5}$ ), we recover the well-known spectrum of energy of one-dimensional Harmonic oscillator.

In what follow, we studied the influence of the modification scalar product on the thermal properties, especially on the specific heat curves, and also on the Fisher information parameter  $F$  for the case of the one-dimensional Harmonic oscillator as a model. This study can be extended for other type of potentials dependent with energy.

(a) first case:  $\nu = 1$ (b) second case:  $\nu = 2$ Figure 2: Specific heat  $C$  versus  $\beta$  in both cases

### III. THE INFLUENCE ON THE THERMAL PROPERTIES OF 1D HARMONIC OSCILLATOR

Lets start with the usual definition of the partition function where

$$Z = \sum_{n=0}^{\infty} e^{-\beta E_n}, \quad (20)$$

In our situation, we can rewritten this equation by

$$Z = \underbrace{\sum_{n=0}^N e^{-\beta E_n}}_{\text{contribution of few levels}} + \underbrace{e^{-\beta \Gamma}}_{\text{term of saturation}}, \quad (21)$$

where  $\Gamma = \frac{1}{|\gamma|}$  for the first choice, and  $\Gamma = \frac{1}{\sqrt{|\gamma|}}$  for the second. with the first term is the contribution of all levels until a start of a saturation behavior, the second is the contribution of saturation of all levels  $n > N$ .

Following the new definition of partition function, we can see that in the limit where  $\gamma \rightarrow 0$  the specific quantum number  $N \rightarrow \infty$ , and so we recover the usual partition function. Moreover, when  $\gamma \rightarrow \infty$ , so  $N \rightarrow 0$ , and consequently  $Z = 1$  which means that all levels are in saturation. Shown in Figure. 2 is the specific heat  $C_v$  versus the inverse of temperature  $\beta$  for different values of  $\gamma$ . Following this figure, one observes that:

- in the limit where  $\gamma$  has a very small values ( $|\gamma| = 10^{-2}$  and  $|\gamma| = 10^{-5}$ ), the curve coincide with the one-dimensional harmonic oscillator in the all range of temperatures. On the other hand, the curves have the same allure as in the case of the one-dimensional harmonic oscillator, but tend slowly to zero in a very low-temperatures ( $\beta \rightarrow \infty$ ) for different values of  $\gamma$ . Now, in the range of high temperatures ( $\beta \rightarrow 0$ ), we have  $C_v \rightarrow 0$ .
- all curves, except the case of the one-dimensional harmonic oscillator, exhibit a transition phase between the growth phase and the phase where the energy shows a saturation.

Finally, we can see that the problem of the wave function with energy-dependent potential leads to the modification of scalar product. This modification affects the spectrum of energy, and consequently the thermal properties of the system in question.

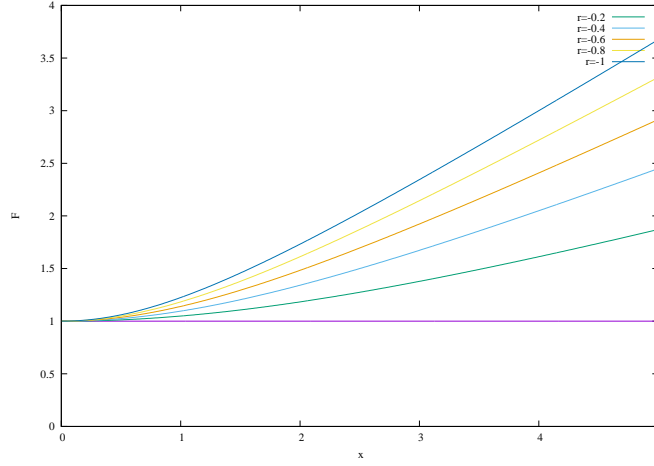


Figure 3: Perey factor  $\mathcal{F}$  for both cases.

### 1. Results and discussions

The present work is devoted to energy dependent potentials which concerns the class of potentials having a coupling constant depending linearly on the energy. In general, the presence of a potential energy dependent in equation wave induces various significant changes to the usual rules of mechanical that quantum of which the most obvious is that the scalar product. the latter is necessary to ensure the conservation of the norm. According this, we have tested, at first, the effect of the modified scalar product on the thermal properties of our system in question through to the spectrum of energy: this modification, as shown in this section, has imposed a constraint on the  $\gamma$  parameter, which leads to the appearance of a phenomenon of saturation in the diagram of spectrum of energy spectrum (see Fig. 1). Then, we have showed that this saturation influence directly the thermal properties of our 1D Harmonic oscillator, especially on the curves of specific heat (see Fig. 2).

We note that the reformulation of the dot product is insufficient to justify the use of other rules that result in an identical manner to quantum mechanics usual. In addition, the testing of these formulas by individual models guarantees not their general validity. This led Formanek et al [6] to propose a quantum theory with the properties similar to those of the ordinary non-relativistic quantum theory. They showed that the wave equation can be transformed into an ordinary Schrodinger equation with a non-local potential. The clarify this, equation (18) can be rewritten by

$$\rho(x) = \left[ \psi^*(x) \left( 1 - \frac{\partial V}{\partial E} \right)^{\frac{1}{2}} \right] \left[ \left( 1 - \frac{\partial V}{\partial E} \right)^{\frac{1}{2}} \psi(x) \right] \equiv \Psi^*(x) \Psi(x), \quad (22)$$

where

$$\Psi = \left( 1 - \frac{\partial V}{\partial E} \right)^{\frac{1}{2}} \psi = \mathcal{F}(x) \psi, \quad (23)$$

and where  $\mathcal{F}(x) = \sqrt{1 - \frac{\partial V}{\partial E}}$  is very similar to the Perey Factor [16–19]: according to the work of Shimizu et al [19], we can attribute the wave function  $\Psi$  to the non-local wave function  $\psi_{NL}$ , and the wave function  $\psi$  to the equivalent local wave function  $\psi_L$ . The ratio  $\left| \frac{\psi_{NL}}{\psi_L} \right|$  forms the Perey Factor: this factor (more precisely, its deviation from unity), is in a sense a measure of the nonlocality of the original interaction. In our case, we see that this factor is greater than unity .

Finally, the non-locality and energy dependence of the potential are known to play an important role in the production of the repulsive behavior in the baryon-baryon scattering at high energies. It is the most important feature of the potential based on the quark model employing the inverse scattering problem [19].

#### IV. THE INFLUENCE ON THE FISHER AND SHANNON PARAMETERS OF QUANTUM INFORMATION

##### A. Fisher parameter

The Fisher information is a quality of an efficient measurement procedure used for estimating ultimate quantum limits. It was introduced by Fisher as a measure of intrinsic accuracy in statistical estimation theory but its basic properties are not completely well known yet, despite its early origin in 1925. Also, it is the main theoretic tool of the extreme physical information principle, a general variational principle which allows one to derive numerous fundamental equations of physics: Maxwell equations, the Einstein field equations, the Dirac and Klein-Gordon equations, various laws of statistical physics and some laws governing nearly incompressible turbulent fluid flows. Fisher information has been very useful and has been applied in different areas: quantum damped harmonic oscillators, D-dimensional Hydrogenic systems, Statistical complexity of  $H^{+2}$ , position-dependent mass oscillators, for the case of Tietz-Wei diatomic molecular model, for the position-dependent mass Schrodinger system, and finally Enhancing quantum Fisher information by utilizing uncollapsing measurements [20–33]

Now, return in our case. The Fisher information of one-dimensional harmonic oscillator with energy-dependent potential, and with corresponding probability density  $\rho$ , is given by

$$F_x = \int \rho(x) \left[ \frac{d \ln \rho(x)}{dx} \right]^2 dx \geq 0, \quad (24)$$

where

$$\rho_n(x, \gamma) = |\psi_{\alpha n}(x, \gamma)|^2 \left( 1 - \frac{dV}{dE} \right) = |\psi_{\alpha n}(x)|^2 f(x), \quad (25)$$

and where with  $f(x) = 1 - \frac{\gamma}{2}x^2$ . By using the following formula

$$\int_{-\infty}^{+\infty} |\psi_{\alpha n}(x)|^2 \left( 1 - \frac{dV}{dE} \right) dx = 1, \quad (26)$$

the constants  $C_{\alpha n}$ , in both cases, are written by

$$C_{\alpha n}^2 = \frac{\sqrt{\lambda_\alpha}}{2^n n! \sqrt{\pi}} \left\{ 1 - \frac{\gamma}{4\lambda_\alpha} (2n+1) \right\}^{-1}. \quad (27)$$

Thus, the corresponding Fisher information written by

$$\begin{aligned} F_x(n, \gamma) &= \int_{-\infty}^{+\infty} \left( 4f\psi'^2 + 4\psi\psi'f' + \psi^2 \frac{f'^2}{f} \right) dx, \\ &= \underbrace{\int_{-\infty}^{+\infty} 4f\psi'^2 dx}_I + \underbrace{\int_{-\infty}^{+\infty} 4\psi\psi'f' dx}_{II} + \underbrace{\int_{-\infty}^{+\infty} \psi^2 \frac{f'^2}{f} dx}_{III}, \end{aligned} \quad (28)$$

where  $\psi' = \frac{d}{dx}\psi$  and  $f' = -\gamma x$ . After evaluating the following terms

$$\psi'_{\alpha n}(x) = C_{\alpha n} e^{-\lambda_\alpha \frac{x^2}{2}} \left\{ -\lambda_\alpha x H_n(\sqrt{\lambda_\alpha} x) + 2n\sqrt{\lambda_\alpha} H_{n-1}(\sqrt{\lambda_\alpha} x) \right\}, \quad (29)$$

$$\begin{aligned} 4f\psi'^2_{\alpha n} &= C_{\alpha n}^2 e^{-\lambda_\alpha x^2} \left\{ 4\lambda_\alpha^2 x^2 H_n(\sqrt{\lambda_\alpha} x)^2 + 16n^2 \lambda_\alpha H_{n-1}(\sqrt{\lambda_\alpha} x)^2 - 16n\lambda_\alpha^{\frac{3}{2}} x H_n(\sqrt{\lambda_\alpha} x) H_{n-1}(\sqrt{\lambda_\alpha} x) \right\}, \\ &+ C_{\alpha n}^2 e^{-\lambda_\alpha x^2} \left\{ -2\gamma \lambda_\alpha^2 x^4 H_n(\sqrt{\lambda_\alpha} x)^2 - 8n^2 \gamma \lambda_\alpha x^2 H_{n-1}(\sqrt{\lambda_\alpha} x)^2 + 8n\gamma \lambda_\alpha^{\frac{3}{2}} x^3 H_n(\sqrt{\lambda_\alpha} x) H_{n-1}(\sqrt{\lambda_\alpha} x) \right\} \end{aligned} \quad (30)$$

$$4\psi\psi'f' = \gamma C_{\alpha n}^2 e^{-\lambda_\alpha x^2} \left\{ 4\lambda_\alpha x^2 H_n(\sqrt{\lambda_\alpha} x)^2 - 8n\sqrt{\lambda_\alpha} x H_n(\sqrt{\lambda_\alpha} x) H_{n-1}(\sqrt{\lambda_\alpha} x) \right\}, \quad (31)$$

$$\psi^2 \frac{(f')^2}{f} = \frac{\gamma^2 C_{\alpha n}^2 e^{-\lambda_\alpha x^2} x^2 H_n^2(\sqrt{\lambda_\alpha} x)}{1 - \frac{\gamma}{2} x^2}, \quad (32)$$

we obtain that

$$I = \frac{4\lambda_\alpha \left(n + \frac{1}{2}\right) - 4n\gamma - \frac{\gamma}{2} \left[(2n+1)^2 + 2\right]}{1 - \frac{\gamma}{4\lambda_\alpha} (2n+1)}, \quad (33)$$

the second

$$II = \frac{2\gamma}{1 - \frac{\gamma}{4\lambda_\alpha} (2n+1)}, \quad (34)$$

$$\begin{aligned} III &= \gamma^2 C_{\alpha n}^2 \int_{-\infty}^{+\infty} \frac{e^{-\lambda_\alpha x^2} x^2 H_n^2(\sqrt{\lambda_\alpha} x)}{1 - \frac{\gamma}{2} x^2} dx = \frac{\gamma^2 C_{\alpha n}^2}{\lambda_\alpha^{\frac{3}{2}}} \int_{-\infty}^{+\infty} \frac{e^{-y^2} y^2 H_n^2(y)}{1 - \frac{\gamma}{2\lambda} y^2} dy. \\ &\simeq \frac{\gamma^2 C_{\alpha n}^2}{\lambda_\alpha^{\frac{3}{2}}} \left\{ \int_{-\infty}^{+\infty} e^{-y^2} y^2 H_n^2(y) dy - \frac{\gamma}{2\lambda} \int_{-\infty}^{+\infty} e^{-y^2} y^4 H_n^2(y) dy \right\} \end{aligned} \quad (35)$$

where  $y = \sqrt{\lambda_\alpha} x$ , and where

$$\frac{1}{1 - \frac{\gamma}{2\lambda} y^2} = \sum_{n=0}^{\infty} (-1)^n \left( \sqrt{\frac{|\gamma|}{2\lambda}} y \right)^{2n} \simeq 1 - \frac{\gamma}{2\lambda} y^2 + \dots \quad (36)$$

So, we have

$$III = \frac{\frac{\gamma^2 \left(n + \frac{1}{2}\right)}{\lambda_\alpha} - \frac{\gamma^3 \left[(2n+1)^2 + 2\right]}{8\lambda_\alpha^2}}{1 - \frac{\gamma}{4\lambda_\alpha} (2n+1)}. \quad (37)$$

Following these results, the final form of Fisher parameter is

$$\begin{aligned} F_\alpha(n, \gamma) &= \left\{ 1 - \frac{\gamma}{4\lambda_\alpha} (2n+1) \right\}^{-1} \times \\ &\quad \left\{ 4\lambda_\alpha \left(n + \frac{1}{2}\right) - 4n\gamma - \frac{\gamma}{2} \left[(2n+1)^2 + 2\right] + 2\gamma + \frac{\gamma^2 \left(n + \frac{1}{2}\right)}{\lambda_\alpha} - \frac{\gamma^3 \left[(2n+1)^2 + 2\right]}{8\lambda_\alpha^2} \right\}. \end{aligned} \quad (38)$$

This formula are plotted in the Figure. 4. This figure showed the parameter of Fisher versus a quantum number for different values of  $\gamma$  in both cases: the case of harmonic oscillator is also inserted for comparison. We observe that the Fisher parameter  $F_\alpha(\alpha = 1, 2)$  increases with  $n$  for different values of  $\gamma$ . This increase indicates that  $\rho_n(x, \gamma)$  becomes more and more localized, increasing the accuracy in predicting the localization of the particle [32]. The comparison with the case of one-dimensional Harmonic oscillator ( $\gamma = 0$ ) showed that for the values of  $|\gamma| < 0.3$ , the curves of the Fisher parameter are below to the case of 1D harmonic oscillator. On the other hand, and from the values of  $|\gamma| > 0.4$ , these curves move above this last.

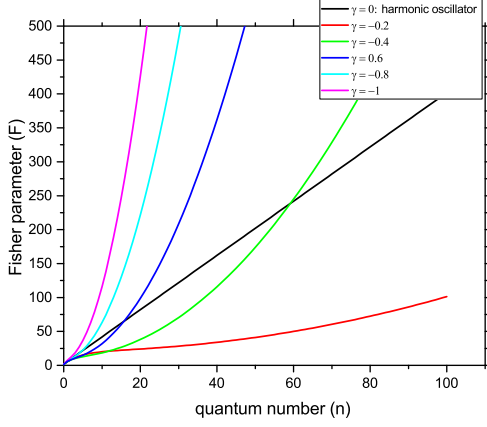
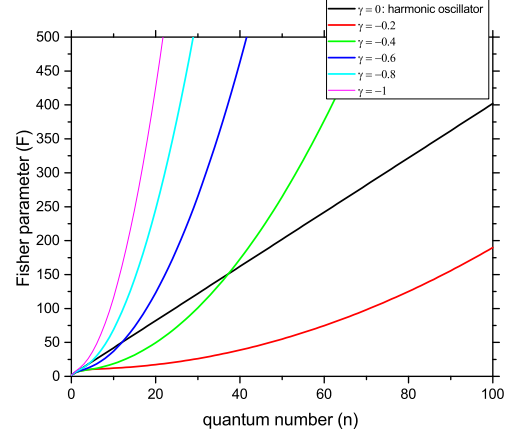
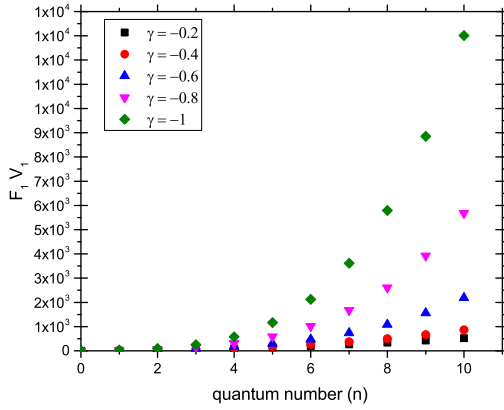
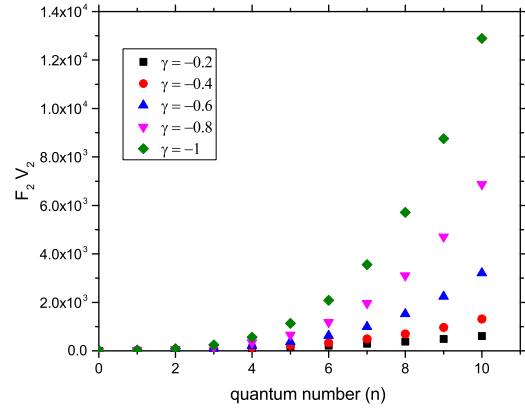
Now, in order to justify our equation for the parameter of Fisher, the well-known Cramer-Rao uncertainty relation

$$F_\alpha \cdot V_\alpha \geq 1, \quad (39)$$

with  $V_\alpha = \langle x^2 \rangle_\alpha - \langle x \rangle_\alpha^2$  must be fulfilled. This parameter is an inequality which involves fisher information and variance. The variance  $V_\alpha$  requires the knowledge of both parameters  $\langle x^2 \rangle_\alpha$  and  $\langle x \rangle_\alpha^2$ . In our case, the modification of the scalar product affects leads to the following equations:

$$\langle x \rangle_\alpha = \int_{-\infty}^{+\infty} x \psi_n^2(x) f(x) dx = 0, \quad (40)$$



(a) first case with  $\nu = 1$ .(b) second case with  $\nu = 2$ .Figure 4: Parameter of Fisher ( $F$ ) versus a quantum number ( $n$ ) for both cases.(a) first case where  $\nu = 1$ .(b) second case where  $\nu = 2$ .Figure 5: The Cramer-Rao uncertainty relation versus quantum number  $n$  for different values of  $\gamma$ .

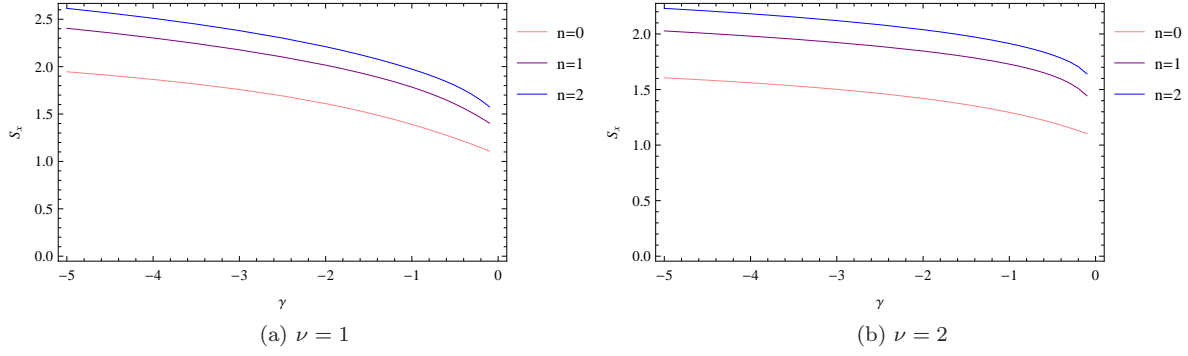
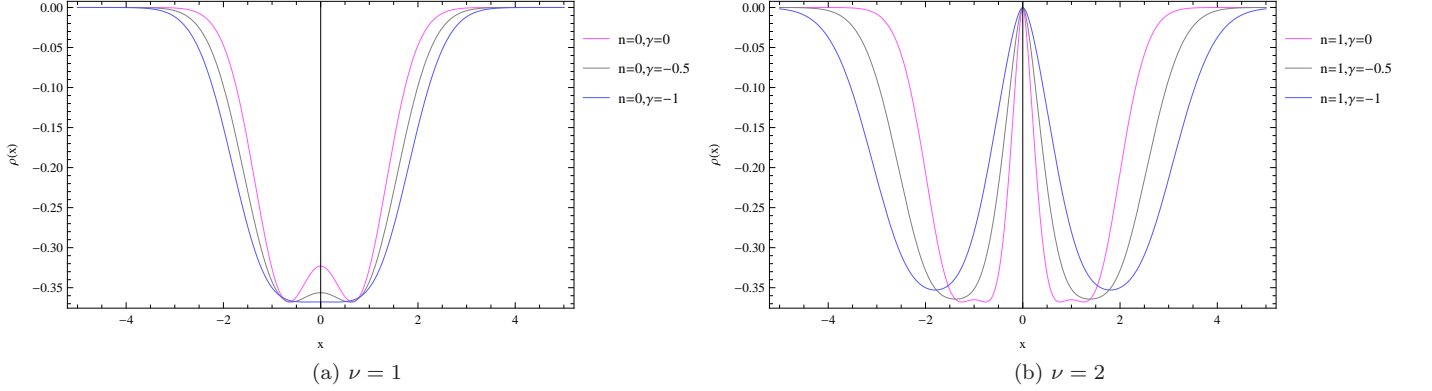
and

$$\langle x^2 \rangle_\alpha = \int_{-\infty}^{+\infty} x^2 \psi_n^2(x) f(x) dx = \frac{2n+1}{2\lambda_n} \left\{ 1 - \frac{\gamma}{4\lambda_n} (2n+1) \right\}^{-1} \left\{ 1 - \frac{\gamma}{4\lambda_n} \frac{[(2n+1)^2 + 2]}{(2n+1)} \right\}. \quad (41)$$

The uncertainty relation of Cramer-Rao is depicted in Figure. 5. From this Figure, we can see that this parameter increases with  $\gamma$ , and the Eq. (39) is well recovered.

## B. Shannon entropy

Entropic measures provide analytic tools to help us to understand correlations in quantum systems. Shannon has introduced entropy to measure the uncertainty. Now, it has become a universal concept in statistical physics. The Shannon entropy has finding applications in several branches of physics because of its possible applications in a wide range of area such as to the nuclei, for the Morse and the Poschl-Teller potentials, in the relationship between the densities of Shannon entropy and Fisher information for atoms and molecules, in the study of the the position

Figure 6: Behavior of  $S_x$  versus  $\gamma$  varying for different values of  $n$ Figure 7: Behavior of  $\rho(x)$  versus the variable  $x$ 

and momentum information-theoretic measures of a D-dimensional particle in a box (see Ref. ([35]) and references therein).

The position space information entropies for the one-dimensional can be calculated by using [36–41]

$$S_x = - \int |\psi(x)|^2 \ln |\psi(x)|^2 dx, \quad (42)$$

with  $\psi(w)$  being the normalized eigenfunction. In our case, the above equation becomes

$$S_x = - \int \rho_n(x, \gamma) \ln \rho_n(x, \gamma) dx, \quad (43)$$

with  $\rho_n(x, \gamma)$  is defined by the equation (25). In general, explicit derivations of the information entropy are quite difficult. In particular, the derivation of analytical expression for the  $S_x$  is almost impossible. We represent the position information entropy densities respectively by

$$\rho_{S_x} = \rho_n(x, \gamma) \ln \rho_n(x, \gamma). \quad (44)$$

We have plotted the results of this section in some figures. In Fig. 6, we consider  $\hbar = 1, m = 1, \omega = 1, \nu = 1, 2$  for difference  $n$ , where we plotted the Shannon information entropy  $S_x$  versus  $\gamma$ . We also plotted these entropy densities for  $\hbar = 1, m = 1, \omega = 1, \nu = 1, 2$  for difference  $n$  and  $\gamma$  in Fig. 7.

### C. Results and discussions

Fisher information is an important concept in quantum estimation and quantum information theory. Its dependence on the probability of density function  $\rho$ , allows us to test the influence of the modified scalar product, in the case of

the problems of the wave function with energy-dependent potentials, on this parameter. The correspondence between a problem of wave function with an ordinary Schrodinger equation with a non-local potential permits us to consider the dependence of this parameter with non-locality of the potential via a coupling constant depending linearly on the energy, i.e., the parameter  $\gamma$ : In this section we shown that this factor, in both cases, depends strongly with  $\gamma$ : thus, the non-locality of the potential has a direct influence on the parameter of Fisher. Moreover, we have used the well-known Cramer-Rao uncertainty relation in order to justify our results concerning the Fisher parameter.

In the same way, this influence is well established for the case of Shannon entropy: the information entropies of the ground and the excited states are versus  $\gamma$  is shown in Figs. 6. Through this parameter, the influence of the modified scalar product on  $S_x$  is very clear for both cases in consideration. The Fig. 7 illustrates the position information entropy densities  $\rho_s(x) = |\psi_{\alpha n}(x, \gamma)|^2 \left(1 - \frac{dV}{dE}\right)$ . They play a role similar to that of the probability density  $\rho(x) = |\psi_{\alpha n}(x, 0)|^2$  in quantum mechanics.

## V. CONCLUSION

The present work is devoted to energy dependent potentials. We have considered the cases of the one-dimensional Harmonic oscillator with energy dependence. This case leads to a coherent theory. As a first result, we show that the energy dependence affects essentially the eigensolutions. Especially, we observe a saturation on the curves of the spectrum of energy. Also, the presence of the energy dependent potential in a wave equation leads to the modification of the scalar product, necessary to ensure the conservation of the norm. So, we have shown that this modification affects directly, (i) the thermal properties of our systems, and (ii) the form of the parameters of Fisher and Shannon. The uncertainty relation of Cramer-Rao is well-established, which means that our calculations on the Fisher parameter are correct.

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